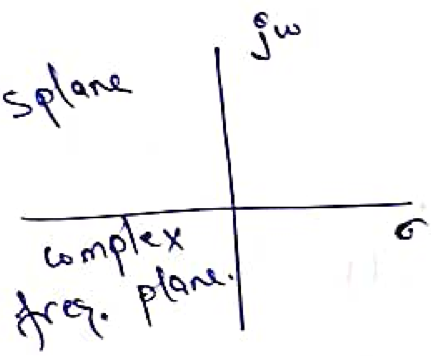
Laplace Transform:-

- laplace transform can be for those signals also which are not Energy or power signal. (here fourier Transform is not used.
- if input & output is given and we require synthesis part.
- it is used to convert time domain signal to freq. domain signal.



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t} \cdot dt$$

Bilateral Laplace Transform

$$\boxed{X(S) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt}$$

- $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t} \cdot dt$

$$X(s) = \int_{-\infty}^{\infty} \underbrace{x(t)\, e^{-\sigma t}}_{x(t)_{f.T}} \cdot e^{-j\omega t} \cdot dt \quad \text{—(1)}$$

from fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

- laplace transform of $x(t)$ is nothing but f.T of signal $x(t) \cdot e^{-\sigma t}$

fourier Transform is nothing but L.T evaluated at $j\omega$ axis.

Ex:- $\cdot\ x(t) = e^{-at}\cdot u(t)$

$$X(s) = \int_{-\infty}^{\infty} \{e^{-at}\cdot u(t)\}\cdot e^{-st}\cdot dt = \frac{1}{s+a}$$

$$L\{e^{-at}(u(t))\} = \frac{1}{s+a}$$

2. $x(t) = -e^{-at}\ u(\cancel{-}t)$

$$X(s) = \frac{1}{s+a}, \quad \text{when } s+a<0$$

to find time domain:-

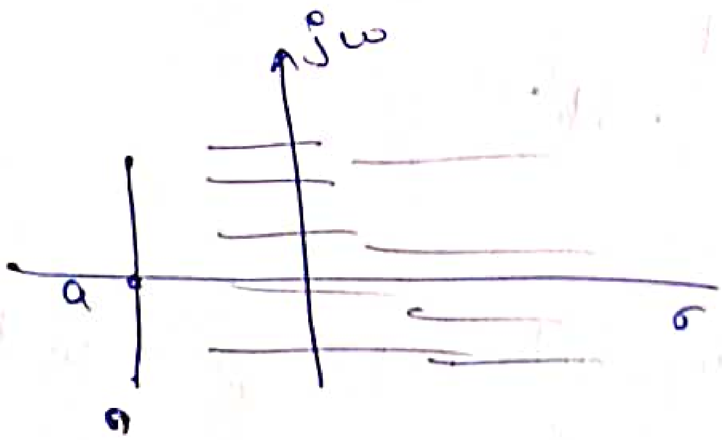
let $x(t) = e^{-at}\cdot u(t)$

$$X(s) = X(\sigma+j\omega) = \int_{-\infty}^{\infty} e^{-at}\cdot e^{-(\sigma+j\omega)t}\cdot dt$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-(\sigma+a)t}}_{+ve}\cdot \underbrace{e^{-j\omega t}}_{\text{Mag}=1}\cdot dt$$

for $X_1(s)$ to converge

$$\sigma+a>0 \Rightarrow \boxed{\sigma>-a}$$



Rational function:-

$f(s) = \frac{N(s)}{D(s)}$, Poles: when $f(s) = \infty$, $D(s) = 0$

Zeroes: when $f(s) = 0$, $N(s) = 0$

Pole $s + a = 0$ {for last quenstion).

Ex: $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} . dt = e^{-st}\Big|_{t=0} = 1.$$

$L\{\delta(t)\} = 1.$

- $x(t) = \delta(t-1)$

$$L\{\delta(t-1)\} = e^{-s}$$

Region of convergence (ROC) of X(S):-

Range of σ [or Re(S)} for which L.T of $x(t)$ converges.

Properties of ROC:-

1. ROC will be a line ||el to $j\omega$ axis.
2. for rational L.T $X(s)$, ROC does not contain any poles.
3. If rational L.T of $\underset{\text{right sided signal}}{x(T)}$ is $X(s)$, then ROC of $X(s)$ will be the region in s-plane, right to the right most pole

$$X(S) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

for

$$\boxed{L\{x^*(t)\} = X^*(s^*)}$$

- $x(t) \longleftrightarrow X(S)$

$$X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \text{, where } \quad S = \sigma + j\omega$$

- $L\{u(t)\} = \frac{1}{s}$

Ex. if $X(S) = \frac{1}{S+4}$.

find causal inverse $x(t)$?

Sol^n $x = e^{-4t} \cdot u(t)$. for causal or $\sigma > -4$

$x = -e^{-4t} \cdot u(t)$ for left side or $\sigma < -4$.

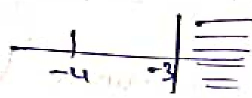
Ex: $\frac{1}{(S+3)(S+4)} = X(S)$

⇒ By partial fraction it can be written as:—

$$X(S) = \frac{1}{S+3} - \frac{1}{S+4}$$

Right most

Case I. if $x(t)$ is right sided signal:—



ROC $= Re(S) > -3$.

$x(t) = e^{-3t} u(t) - e^{-4t} \cdot u(t)$

Case II if $x(t)$ is left side, $ROC = Re(s) < -4$ s.

$$x(t) = -e^{-3t}u(-t) - (-e^{-4t} \cdot u(-t))$$

$$x(t) = e^{-4t}(-t) - e^{-3t}u(-t)$$

Case III :- if $x(t)$ is double sided signal:-

$-4 < \sigma < -3$

$$x(t) = -e^{-3t}u(-t) - e^{-4t}u(t)$$

* $x(t) \xleftrightarrow{F.T} X(j\omega)$

$$IFT(X(j\omega)) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$X(\sigma+j\omega) = X(s) = L.T\{x(t)\} = F.T\{x(t) \cdot e^{-\sigma T}\}$$

$$x(t) \cdot e^{-\sigma t} \longleftrightarrow X(\sigma+j\omega)$$

$$IFT[X(\sigma+j\omega)] = x(t) \cdot e^{-\sigma T} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma+j\omega) \cdot e^{(\sigma+j\omega)t} \cdot d\omega$$

let $\sigma+j\omega = s$

$\frac{ds}{d\omega} = j \Rightarrow d\omega = \frac{ds}{j}$ when $\omega \to -\infty$, $s = \sigma - j\infty$
when $\omega \to \infty$, $s = \sigma + j\infty$

now

$$x(t) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\cdot e^{st}\cdot\frac{ds}{j}$$

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds} \rightarrow \text{laplace inverse}$$

5.

Properties of laplace Transform:—

(1) linearity:—

$$x_1(t) \longleftrightarrow X_1(s), \quad ROC: R_1$$
$$x_2(t) \longleftrightarrow X_2(s), \quad ROC; R_2$$

then

$$\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X_1(s) + \beta X_2(s)$$

ROC: $R_1 \cap R_2$

(2) Time shifting property:—

$$x(t) \longleftrightarrow X(s), \quad ROC: R$$
$$x(t-t_0) \longleftrightarrow e^{-st_0}\cdot X(s), \quad ROC: \text{containing } R$$

(3) Time reversal property:—

$$x(t) \longleftrightarrow X(s); \quad ROC: R$$
$$x(-t) \longleftrightarrow X(-s); \quad ROC: \frac{1}{R}$$

if ROC $R \Rightarrow \sigma > -4$

$\frac{1}{R} = \sigma < 4$

Time scaling property:—

$$x(t) \longleftrightarrow X(s); \quad ROC\ R$$

$$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} \cdot X\left(\frac{s}{\alpha}\right), \quad ROC: \alpha \cdot R$$

5. frequency shifting property:—

$$x(t) \longleftrightarrow x(s), \quad ROC: R$$

$$e^{s_0 t} \cdot x(t) \longleftrightarrow X(s-s_0), \quad ROC: R + Re\{s_0\}.$$

6. Convolution in time property:—

$$x_1(t) \longleftrightarrow X_1(s), \quad ROC: R_1$$

$$x_2(t) \longleftrightarrow X_2(s), \quad ROC; R_2$$

$$x_1(t) * x_2(t) \longleftrightarrow x_1(s) \cdot X_2(s), \quad ROC: R_1 \cap R_2$$

7. Differentiation in time:—

$$x(t) \longleftrightarrow X(s), \quad ROC: R \qquad \text{for limit } \int^{\infty}$$

$$\frac{d}{dt}(x(t)) \longleftrightarrow s \cdot X(s), \quad ROC: \text{containing } R$$

$$\frac{d^2}{dt^2}\{x(t)\} \longleftrightarrow s^2 X(s) - s\,X(0) - x'(0)$$

8). Differentiation in frequency domain:—

$$x(t) \longleftrightarrow X(s), \quad ROC: R$$

$$-t \cdot x(t) \longleftrightarrow \frac{d}{ds} X(s), \quad ROC: R$$

$$\cancel{t^n x(t) \longleftrightarrow} \quad t^n e^{-at} u(t) \longleftrightarrow \overset{*}{\frac{n!}{(s+a)^{n+1}}} \qquad \sigma > -a$$

9. Integration in time Domain:—

$$x(t) \longleftrightarrow X(s), \quad ROC: R$$

$$\int_{-\infty}^{t} x(\tau)\cdot d\tau \longleftrightarrow \frac{X(s)}{s}, \quad ROC: R \cap \{Re\{s\} > 0\}$$

10. Integration in frequency Domain (Division by t property):

$$x(t) \longleftrightarrow X(s), \quad ROC: R$$

$$\frac{x(t)}{t} \longleftrightarrow \int_{s}^{\infty} X(s)\,ds, \quad ROC: R$$

11. Initial and final value theorems:—

initial value theorem:—

(i) $x(t) = 0$ for $t < 0$, $x(t) = x_1(t)\cdot u(t)$

(ii) must not contain impulses or its higher derivatives.

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} s\,X(s)$$

if $X(s)$ is a rational fun^n $\quad X(s) = \frac{N(s)}{D(s)}$

(i) if $N(s) > D(s)$ (in degree term)

we can not find initial value.

(ii) Deg. of $D(s) > N(s)$ to find initial value.

(iii) if [degree of $D(s)$ − Deg. of $N(s)$] > 1

then

$$x(0^+) = 0.$$

final value theorem:—

- $x(t) = 0 \ , \ t < 0$
- $x(t)$ must not contain impulse or its higher derivatives
- poles of $sX(s)$ must lie in LHS of s-plane.
- we can not find final value for periodic and unbounded function.

$$\lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} sX(s)$$

Some important results:—

signal $x(t)$		$X(s)$	ROC : R
$\delta(t)$	$\longleftrightarrow$	1	Entire s-plane
$u(t)$	$\longleftrightarrow$	$\frac{1}{s}$,	$Re\{s\} > 0$
$-u(-t)$	$\longleftrightarrow$	$\frac{1}{s}$,	$Re(s) < 0$
$e^{-at}u(t)$	$\longleftrightarrow$	$\frac{1}{s+a}$,	$Re(s) > -a$
$-e^{-at}u(-t)$	$\longleftrightarrow$	$\frac{1}{s+a}$,	$Re\{s\} < -a$
$t^n.u(t)$	$\longleftrightarrow$	$\frac{n!}{(s)^{n+1}}$,	$Re\{s\} > 0$
$-t^n u(-t)$	$\longleftrightarrow$	$\frac{n!}{(s)^{n+1}}$,	$Re(s) < 0$
$t^n . e^{-at} u(t)$	$\longleftrightarrow$	$\frac{n!}{(s+a)^{n+1}}$,	$Re(s) > -a$

$-t^n e^{-at} \cdot u(-t) \longleftrightarrow \dfrac{n!}{(s+a)^{n+1}}$, $Re(s) < -a$

$\cos \omega_0 t \cdot u(t) \longleftrightarrow \dfrac{s}{s^2+\omega_0^2}$, $Re(s) > 0$

$\sin \omega_0 t \cdot u(t) \longleftrightarrow \dfrac{\omega_0}{s^2+\omega_0^2}$, $Re(s) > 0$

$e^{-at} \cos \omega_0 t \cdot u(t) \longleftrightarrow \dfrac{s+a}{(s+a)^2+\omega_0^2}$, $Re\{s\} > -a$

$e^{-at} \cdot \sin \omega_0 t \cdot u(t) \longleftrightarrow \dfrac{\omega_0}{(s+a)^2+\omega_0^2}$, $Re\{s\} > -a$

Causality & stability:—

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

Causal:— if $h(t) = 0$ for $t < 0$

non Causal:— $h(t) \neq 0$ for $t = 0$

anticausal:— $h(t) = 0$, for $t > 0$

- if a system is causal, then ROC of its system f^n must be right side to the right most pole. but converse is not true.

system funⁿ is rational, ROC is right to right most pole, then system will be causal.

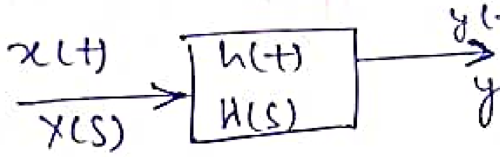
<u>stability</u> :—

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

• if $h(t) \longleftrightarrow H(S)$, ROC : R {it should include $j\omega$ axis}

$j\omega$ - axis must be included in ROC of its system func"

• imp.



$$y(t) = x(t) * h(t)$$
$$Y(S) = X(S) \cdot H(S)$$

$$\therefore H(S) = \frac{Y(S)}{X(S)}$$

Ex: if i/p. o/p relation of an LTI system is given by a differential eqⁿ as:-

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

find all possible impulse responses and comment on stability of the system.

ans taking L.T both side

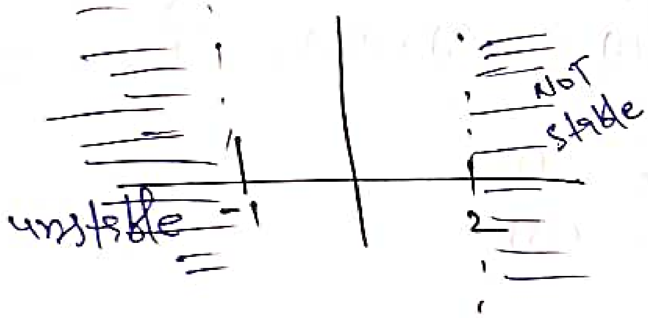
$$s^2 y(s) - s\,y(s) - 2y(s) = x(s)$$

$$y(s)(s^2 - s - 2) = x(s)$$

$$\frac{y(s)}{x(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s+1)(s-2)} = H(s)$$

$$H(s) = \frac{1}{(s+1)(s-2)} = \frac{-1}{3(s+1)} + \frac{1}{3(s-2)}$$

Case I. if h(t) is Right sided:-



$$h(t) = -\frac{1}{3} e^{-t} \cdot u(t) + \frac{1}{3} e^{2t} u(t)$$

unstable

Re(s) > 2

Case II if h(t) is left sided:-

$$h(t) = (-1) - \frac{1}{3} e^{-t} u(-t) + \frac{1(-1)}{3} e^{2t} u(-t)$$, Re(s) < -1

unstable

Case 3: -1 < h(t) < 2 (lies) doublesided.

-1 < Re(s) < 2

$$h(t) = -\frac{1}{3} e^{-t} u(t) + \frac{1}{3}(-1) e^{2t} u(-t)$$ stable

inverse:—

$$F(s) = \frac{1}{s+a}, \quad Re(s) > -a$$

$$f(t) = e^{-at} u(t)$$

(ii) $$F(s) = \frac{n!}{(s+a)^{n+1}}$$

$$f(t) = t^n e^{-at} u(t)$$

Minimum phase system:—

- All poles and zeroes must be located in LHS of s plane, so that the inverse of the system will also be causal & stable.

Ex: $$H(s) = \frac{(s+2)(s+3)}{(s+1)(s+2)(s+5)}$$ = minimum phase system

$$H(s) = \frac{(s-2)(s+3)}{(s+1)(s+2)(s+5)}$$ = mixed phase system

$$H(s) = \frac{(s-2)(s-3)}{(s+1)(s+2)(s+5)}$$ = max. phase system.

- Total no. of poles = total no. of zeroes = 3
- No of zeroes in finite s-plane $z = 2$